

DIMENSION OF ZERO WEIGHT SPACE: AN ALGEBRO-GEOMETRIC APPROACH

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1. INTRODUCTION

Let G be a connected, adjoint, simple algebraic group over the complex numbers \mathbb{C} with a maximal torus T and a Borel subgroup $B \supset T$. The study of zero weight spaces in irreducible representations of G has been a topic of considerable interest; there are many works which study the zero weight space as a representation space for the Weyl group. In this paper, we study the variation on the dimension of the zero weight space as the irreducible representation varies over the set of dominant integral weights for T which are lattice points in a certain polyhedral cone.

The theorem proved here asserts that the zero weight spaces have dimensions which are piecewise polynomial functions on the polyhedral cone of dominant integral weights. The precise statement of the theorem is given below.

Let $\Lambda = \Lambda(T)$ be the character group of T and let $\Lambda^+ \subset \Lambda$ (resp. Λ^{++}) be the semigroup of dominant (resp. dominant regular) weights. Then, by taking derivatives, we can identify Λ with Q , where Q is the root lattice (since G is an adjoint group). For $\lambda \in \Lambda^+$, let $V(\lambda)$ be the irreducible G -module with highest weight λ . Let $\mu_0 : \Lambda^+ \rightarrow \mathbb{Z}_+$ be the function: $\mu_0 = \dim V(\lambda)_0$, where $V(\lambda)_0$ is the 0-weight space of $V(\lambda)$.

Let $\Gamma = \Gamma_G \subset Q$ be the sublattice as in Theorem (3.1).

Also, let $\Lambda(\mathbb{R}) := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ and let $\Lambda^{++}(\mathbb{R})$ be the cone inside $\Lambda(\mathbb{R})$ generated by Λ^{++} . Let $C_1, \dots, C_N \subset \Lambda^{++}(\mathbb{R})$ be the chambers (i.e., the GIT classes in $\Lambda^{++}(\mathbb{R})$ of maximal dimension: equal to the dimension of $\Lambda(\mathbb{R})$, with respect to the T -action) (see Section 2).

For any $w \in W$ and $1 \leq i \leq \ell$, define the hyperplane

$$H_{w,i} := \{\lambda \in \Lambda(\mathbb{R}) : \lambda(wx_i) = 0\},$$

where W is the Weyl group of G and $\{x_1, \dots, x_\ell\}$ is the basis of \mathfrak{t} dual to the basis of \mathfrak{t}^* given by the simple roots. Then, by virtue of Corollary (3.6), C_1, \dots, C_N are the connected components of

$$\Lambda^{++}(\mathbb{R}) \setminus \left(\bigcup_{w \in W, 1 \leq i \leq \ell} H_{w,i} \right).$$

With this notation, we have the following main result of our paper (cf. Theorem (4.1)).

Theorem (1.1). *Let $\bar{\mu} = \mu + \Gamma$ be a coset of Γ in Q . Then, for any GIT class C_k , $1 \leq k \leq N$, there exists a polynomial $f_{\bar{\mu},k} : \Lambda(\mathbb{R}) \rightarrow \mathbb{R}$ with rational coefficients of degree $\leq \dim_{\mathbb{C}} X - \ell$, such that*

$$(1) \quad f_{\bar{\mu},k}(\lambda) = \mu_0(\lambda), \quad \text{for all } \lambda \in \bar{C}_k \cap \bar{\mu},$$

where \bar{C}_k is the closure of C_k inside $\Lambda(\mathbb{R})$ and X is the full flag variety G/B . Further, $f_{\bar{\mu},k}$ has constant term 1.

The proof of the above theorem relies on Geometric Invariant Theory (GIT). Specifically, we realize the function μ_0 restricted to $\bar{C}_k \cap \Lambda$ as an Euler-Poincaré characteristic of a reflexive sheaf on a certain GIT quotient (depending on C_k) of $X = G/B$ via the maximal torus T . Then, one can use the Riemann-Roch theorem for singular varieties to calculate this Euler-Poincaré characteristic. From this calculation, we conclude that the function μ_0 restricted to $\bar{C}_k \cap (\mu + \Gamma)$ is a polynomial function. The result on descent of the homogeneous line bundles on X to the GIT quotient plays a crucial role (cf. Lemma (3.7)).

We end the paper by determining these piecewise polynomials for the groups of type A_2 and B_2 (in Section 5) and A_3 (in Section 6), all of which we do via some well-known branching laws.

The results of the paper can easily be extended to show the piecewise polynomial behavior of the dimension of any weight space (of a fixed weight μ) in any finite dimensional irreducible representation $V(\lambda)$.

By a similar proof, we can also obtain a piecewise polynomial behavior of the dimension of H -invariant subspace in any finite dimensional irreducible representation $V(\lambda)$ of G , where $H \subset G$ is a reductive subgroup. However, the results in this general case are not as precise (cf. Remark (4.2)).

It should be mentioned that Meinrenken-Sjamaar [MS] have obtained a result similar to our above result Theorem (1.1) (also in the generality of H -invariants) by using techniques from Symplectic Geometry. But, their result in the case of T -invariants is less precise than our Theorem (1.1).

The example of PGL_4 suggests that the part of our theorem describing the domains of validity of these piecewise polynomial functions is not optimal. Moreover, our theorem says nothing about the explicit nature of these polynomials. So this work should be taken only as a step towards the eventual goal of describing the variation of the dimension of the zero weight space as the irreducible representation varies over the set of dominant integral weights for T .

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2. NOTATION

Let G be a connected, adjoint, semisimple algebraic group over the complex numbers \mathbb{C} . Fix a Borel subgroup B and a maximal torus $T \subset B$. We denote their Lie algebras by the corresponding Gothic characters: \mathfrak{g} , \mathfrak{b} and \mathfrak{t} respectively. Let $R^+ \subset \mathfrak{t}^*$ be the set of positive roots (i.e., the roots of B) and let $\Delta = \{\alpha_1, \dots, \alpha_\ell\} \subset R^+$ be the set of simple roots. Let $Q = \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i$ be the root lattice. Then, the group of characters Λ of T can be identified with Q (since G is adjoint) by taking the derivative. We will often make this identification. Let Λ^+ (resp. Λ^{++}) be the semigroup of dominant (resp. dominant regular) weights, i.e.,

$$\Lambda^+ := \{\lambda \in \Lambda : \lambda(\alpha_i^\vee) \in \mathbb{Z}_+, \text{ for all the simple coroots } \alpha_i^\vee\},$$

and

$$\Lambda^{++} := \{\lambda \in \Lambda^+ : \lambda(\alpha_i^\vee) \geq 1 \text{ for all } \alpha_i^\vee\}.$$

Then, Λ^+ bijectively parameterizes the isomorphism classes of finite dimensional irreducible G -modules. For $\lambda \in \Lambda^+$, let $V(\lambda)$ be the corresponding irreducible G -module (with highest weight λ).

Let $W := N(T)/T$ be the Weyl group of G , where $N(T)$ is the normalizer of T in G . Let $\Lambda(\mathbb{R}) := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ and let $\Lambda^+(\mathbb{R})$ (resp. $\Lambda^{++}(\mathbb{R})$) be the cone inside $\Lambda(\mathbb{R})$ generated by Λ^+ (resp. Λ^{++}). Any element $\lambda \in \Lambda(\mathbb{R})$ can uniquely be written as

$$(2) \quad \lambda = \sum_{i=1}^{\ell} z_i \omega_i, \quad z_i \in \mathbb{R},$$

where $\omega_i \in \Lambda^+(\mathbb{R})$ is the i -th fundamental weight:

$$\omega_i(\alpha_j^\vee) = \delta_{i,j}.$$

Then,

$$\Lambda^+(\mathbb{R}) = \bigoplus_{i=1}^{\ell} \mathbb{R}_{\geq 0} \omega_i, \quad \Lambda^{++}(\mathbb{R}) = \bigoplus_{i=1}^{\ell} \mathbb{R}_{> 0} \omega_i,$$

where $\mathbb{R}_{\geq 0}$ (resp. $\mathbb{R}_{> 0}$) is the set of non-negative (resp. strictly positive) real numbers. We will denote any $\lambda \in \Lambda(\mathbb{R})$ in the coordinates $z_\lambda = (z_i)_{1 \leq i \leq \ell}$ as in (2).

A function $f : S \subset \Lambda^+ \rightarrow \mathbb{Q}$ defined on a subset S of Λ^+ is called a *polynomial function* if there exists a polynomial $\hat{f}(z) \in \mathbb{Q}[z_i]_{1 \leq i \leq \ell}$ such that $f(\lambda) = \hat{f}(z_\lambda)$, for all $\lambda \in S$.

For any $\lambda \in \Lambda$, we have the G -equivariant line bundle $\mathcal{L}(\lambda)$ on $X := G/B$ associated to the principal B -bundle $G \rightarrow G/B$ via the character λ^{-1} of B , i.e.,

$$\mathcal{L}(\lambda) = G \times^B \mathbb{C}_{-\lambda} \rightarrow G/B,$$

where $\mathbb{C}_{-\lambda}$ denotes the one dimensional T -module with weight $-\lambda$. (Observe that for any $\lambda \in \Lambda$, the T -module structure on $\mathbb{C}_{-\lambda}$ extends to a B -module structure). The line bundle $\mathcal{L}(\lambda)$ is ample if and only if $\lambda \in \Lambda^{++}$.

Following Dolgachev-Hu [DH], $\lambda, \mu \in \Lambda^{++}(\mathbb{R})$ are said to be *GIT equivalent* if $X^{ss}(\lambda) = X^{ss}(\mu)$, where $X^{ss}(\lambda)$ denotes the set of semistable points in X with respect to the element $\lambda \in \Lambda^{++}(\mathbb{R})$. Recall that if $\lambda \in \Lambda^{++}(\mathbb{Q}) := \bigoplus_{i=1}^{\ell} \mathbb{Q}_{>0} \omega_i$, then $X^{ss}(\lambda)$ is the set of T -semistable points of X with respect to the T -equivariant line bundle $\mathcal{L}(d\lambda)$, for any positive integer d such that $d\lambda \in \Lambda^{++}$.

Definition (2.1). By a *rational polyhedral cone* C in $\Lambda^{++}(\mathbb{R})$, one means a subset of $\Lambda^{++}(\mathbb{R})$ defined by a finite number of linear inequalities with rational coefficients.

For a \mathbb{R} -linear form f on $\Lambda(\mathbb{R})$ which is non-negative on C , the set of points $c \in C$ such that $f(c) = 0$ is called a *face* of C .

By [DH] or [R, Proposition 7], any GIT equivalence class in $\Lambda^{++}(\mathbb{R})$ is the relative interior of a rational polyhedral cone in $\Lambda^{++}(\mathbb{R})$ and moreover there are only finitely many GIT classes (cf. [DH, Theorem 1.3.9] or [R, Theorem 3]). Let C_1, \dots, C_N be the GIT classes of maximal dimension, i.e., of dimension equal to that of $\Lambda(\mathbb{R})$. These are called *chambers*. Let $X_T(C_k)$ denote the GIT quotient $X^{ss}(\lambda)/T$ for any $\lambda \in C_k$.

Since for any $\lambda \in \Lambda^+$, the irreducible module $V(\lambda)$ has its zero weight space $V(\lambda)_0$ nonzero, we have $X^{ss}(\lambda) \neq \emptyset$ for any $\lambda \in \Lambda^{++}(\mathbb{R})$.

Let $\mathfrak{t}_+ := \{x \in \mathfrak{t} : \alpha_i(x) \geq 0, \text{ for all the simple roots } \alpha_i\}$ be the dominant chamber. Clearly,

$$(3) \quad \mathfrak{t}_+ = \bigoplus_{i=1}^{\ell} \mathbb{R}_+ x_i,$$

where $\{x_i\}$ is the basis of \mathfrak{t} dual to the basis of \mathfrak{t}^* consisting of the simple roots, i.e.,

$$(4) \quad \alpha_i(x_j) = \delta_{i,j}.$$

3. DESCENT OF LINE BUNDLES TO GIT QUOTIENTS AND DETERMINATION OF CHAMBERS

There exists the largest lattice $\Gamma \subset Q$ such that for any $\lambda \in \Lambda^{++} \cap \Gamma$, the homogeneous line bundle $\mathcal{L}(\lambda)$ descends as a line bundle $\widehat{\mathcal{L}}(\lambda)$ on the GIT quotient $X_T(\lambda)$. In fact, Γ is determined precisely in [Ku, Theorem 3.10] for any simple G , which we recall below.

Theorem (3.1). *For any simple G , $\Gamma = \Gamma_G$ is the following lattice (following the indexing in [B, Planche I-IX]).*

- (1) G of type A_ℓ ($\ell \geq 1$) : Q
- (2) G of type B_ℓ ($\ell \geq 3$) : $2Q$
- (3) G of type C_ℓ ($\ell \geq 2$) : $\mathbb{Z}2\alpha_1 + \cdots + \mathbb{Z}2\alpha_{\ell-1} + \mathbb{Z}\alpha_\ell$
- (4) G of type D_4 : $\{n_1\alpha_1 + 2n_2\alpha_2 + n_3\alpha_3 + n_4\alpha_4 : n_i \in \mathbb{Z} \text{ and } n_1 + n_3 + n_4 \text{ is even}\}$
- (5) G of type D_ℓ ($\ell \geq 5$) :
 $\{2n_1\alpha_1 + 2n_2\alpha_2 + \cdots + 2n_{\ell-2}\alpha_{\ell-2} + n_{\ell-1}\alpha_{\ell-1} + n_\ell\alpha_\ell : n_i \in \mathbb{Z} \text{ and } n_{\ell-1} + n_\ell \text{ is even}\}$
- (6) G of type G_2 : $\mathbb{Z}6\alpha_1 + \mathbb{Z}2\alpha_2$
- (7) G of type F_4 : $\mathbb{Z}6\alpha_1 + \mathbb{Z}6\alpha_2 + \mathbb{Z}12\alpha_3 + \mathbb{Z}12\alpha_4$
- (8) G of type E_6 : $6\widetilde{\Lambda}$
- (9) G of type E_7 : $12\widetilde{\Lambda}$
- (10) G of type E_8 : $60Q$,

where $\widetilde{\Lambda}$ is the lattice generated by the fundamental weights.

Definition (3.2). Let S be any connected reductive algebraic group acting on a projective variety \mathbb{X} and let \mathbb{L} be an S -equivariant line bundle on \mathbb{X} . Let $O(S)$ be the set of all one parameter subgroups (for short OPS) in S . Take any $x \in \mathbb{X}$ and $\delta \in O(S)$. Then, \mathbb{X} being projective, the morphism $\delta_x : \mathbb{G}_m \rightarrow X$ given by $t \mapsto \delta(t)x$ extends to a morphism $\widetilde{\delta}_x : \mathbb{A}^1 \rightarrow X$. Following Mumford, define a number $\mu^\mathbb{L}(x, \delta)$ as follows: Let $x_o \in X$ be the point $\widetilde{\delta}_x(0)$. Since x_o is \mathbb{G}_m -invariant via δ , the fiber of \mathbb{L} over x_o is a \mathbb{G}_m -module; in particular, it is given by a character of \mathbb{G}_m . This integer is defined as $\mu^\mathbb{L}(x, \delta)$.

Let V be a finite dimensional representation of S and let $i : \mathbb{X} \hookrightarrow \mathbb{P}(V)$ be an S -equivariant embedding. Take $\mathbb{L} := i^*(O(1))$. Let $\lambda \in O(S)$ and let $\{e_1, \dots, e_n\}$ be a basis of V consisting of eigenvectors, i.e., $\lambda(t) \cdot e_l = t^{\lambda_l} e_l$, for $l = 1, \dots, n$. For any $x \in \mathbb{X}$, write $i(x) = [\sum_{l=1}^n x_l e_l]$. Then, it is easy to see that, we have ([MFK, Proposition 2.3, page 51])

$$(5) \quad \mu^\mathbb{L}(x, \lambda) = \max_{l: x_l \neq 0} (-\lambda_l).$$

We record the following standard properties of $\mu^\mathbb{L}(x, \delta)$ (cf. [MFK, Chap. 2, §1]):

Proposition (3.3). *For any $x \in \mathbb{X}$ and $\delta \in O(S)$, we have the following (for any S -equivariant line bundles $\mathbb{L}, \mathbb{L}_1, \mathbb{L}_2$):*

- (a) $\mu^{\mathbb{L}_1 \otimes \mathbb{L}_2}(x, \delta) = \mu^{\mathbb{L}_1}(x, \delta) + \mu^{\mathbb{L}_2}(x, \delta)$.
- (b) *If $\mu^{\mathbb{L}}(x, \delta) = 0$, then any element of $H^0(\mathbb{X}, \mathbb{L})^S$ which does not vanish at x does not vanish at $\lim_{t \rightarrow 0} \delta(t)x$ as well.*
- (c) *For any projective S -variety \mathbb{X}' together with an S -equivariant morphism $f : \mathbb{X}' \rightarrow \mathbb{X}$ and any $x' \in \mathbb{X}'$, we have $\mu^{f^*\mathbb{L}}(x', \delta) = \mu^{\mathbb{L}}(f(x'), \delta)$.*
- (d) *(Hilbert-Mumford criterion) Assume that \mathbb{L} is ample. Then, $x \in \mathbb{X}$ is semistable (resp. stable) (with respect to \mathbb{L}) if and only if $\mu^{\mathbb{L}}(x, \delta) \geq 0$ (resp. $\mu^{\mathbb{L}}(x, \delta) > 0$), for all non-constant $\delta \in O(S)$.*

Lemma (3.4). *For any $\lambda \in \Lambda^{++}$, the set $X^s(\lambda)$ of stable points (in $X^{ss}(\lambda)$) is nonempty.*

Proof. Consider the embedding

$$i_\lambda : X \hookrightarrow \mathbb{P}(V(\lambda)), \quad gB \mapsto [gv_\lambda],$$

where v_λ is a highest weight vector in $V(\lambda)$. Then, the line bundle $\mathcal{O}(1)$ over $\mathbb{P}(V(\lambda))$ restricts to the line bundle $\mathcal{L}(\lambda)$ on X via i_λ (as can be easily seen).

Consider the open subset $U_\lambda \subset X$ defined by $U_\lambda = \{gB \in X : gv_\lambda \text{ has a nonzero component in each of the weight spaces } V(\lambda)_{w\lambda} \text{ of weight } w\lambda, \text{ for all } w \in W\}$.

Since $V(\lambda)$ is an irreducible G -module, it is easy to see that U_λ is nonempty. We claim that

$$(6) \quad U_\lambda \subset X^s(\lambda).$$

By the Hilbert-Mumford criterion (cf. Proposition (3.3) (d)), it suffices to prove that for any $gB \in U_\lambda$, the Mumford index

$$(7) \quad \mu^{\mathcal{L}(\lambda)}(gB, \sigma) > 0,$$

for any nonconstant one parameter subgroup $\sigma : \mathbb{G}_m \rightarrow T$. Express

$$gv_\lambda = \sum_{\mu \in X(T)} v_\mu,$$

as a sum of weight vectors. Let $\dot{\sigma}$ be the derivative of σ considered as an element of \mathfrak{t} . Then, by the identity (5),

$$(8) \quad \begin{aligned} \mu^{\mathcal{L}(\lambda)}(gB, \sigma) &= \max_{\substack{\mu \in X(T): \\ v_\mu \neq 0}} \{-\mu(\dot{\sigma})\} \\ &\geq \max_{w \in W} \{\lambda(-w\dot{\sigma})\}, \text{ since } gB \in U_\lambda. \end{aligned}$$

Choose $w' \in W$ such that $-w'\dot{\sigma} \in \mathfrak{t}_+$. Since σ is nonconstant, $-w'\dot{\sigma} \neq 0$. We next claim that

$$(9) \quad \lambda(-w'\dot{\sigma}) > 0 :$$

To prove this, first observe that any fundamental weight ω_j belongs to $\oplus_{i=1}^{\ell} \mathbb{Q}_{>0} \alpha_i$. (One could check this case by case for any simple group from [B, Planche I-IX]. Alternatively, one can give a uniform proof as well.) Thus, by the decomposition (3), since $-\omega' \dot{\sigma} \neq 0 \in \mathfrak{t}_+$, we get (9). In particular, by (8), $\mu^{\mathcal{L}(\lambda)}(gB, \sigma) > 0$, proving (7). This proves the lemma. \square

Proposition (3.5). *For $\lambda \in \Lambda^{++}$, $X^s(\lambda) \neq X^{ss}(\lambda)$ if and only if there exists $w \in W$ and x_j such that $\lambda(wx_j) = 0$, where $x_i \in \mathfrak{t}$ is defined by (4).*

Proof. Assume first that $X^s(\lambda) \neq X^{ss}(\lambda)$. Take $x \in X^{ss}(\lambda) \setminus X^s(\lambda)$. Then, by the Mumford criterion Proposition (3.3) (d), there exists a non-constant one parameter subgroup δ in T such that $\mu^{\mathcal{L}(\lambda)}(x, \delta) = 0$. Since both of $X^s(\lambda)$ and $X^{ss}(\lambda)$ are $N(T)$ -stable under the left multiplication on X by $N(T)$ (by loc. cit.), we can assume that δ is G -dominant, i.e., the derivative $\dot{\delta} \in \mathfrak{t}_+$. Thus, by Proposition (3.3) (b), $x_\delta := \lim_{t \rightarrow 0} \delta(t)x \in X^{ss}(\lambda)$, since x is semistable. Let G^δ be the fixed point subgroup of G under the conjugation action by δ . Then, G^δ is a (connected) Levi subgroup of G . Let W^{G^δ} be the set of minimal length coset representatives in the cosets W/W_{G^δ} , where $W_{G^\delta} \subset W$ is the Weyl group of G^δ . The fixed point set of X under the left multiplication by δ is given by $X^\delta = \sqcup_{v \in W_{G^\delta}} G^\delta v^{-1}B/B$. Let $w \in W^{G^\delta}$ be such that $x_\delta \in G^\delta w^{-1}B/B$. Thus, by [Ku, Lemma 3.4],

$$(10) \quad w^{-1}\lambda \in \sum_{\alpha_i \in \Delta(G^\delta)} \mathbb{Z}\alpha_i,$$

where $\Delta(G^\delta) \subset \Delta$ is the set of simple roots of G^δ . Since δ is non-constant, G^δ is a proper Levi subgroup. Take $\alpha_j \in \Delta \setminus \Delta(G^\delta)$. Then, by (10), $\lambda(wx_j) = 0$.

Conversely, assume that

$$(11) \quad \lambda(wx_j) = 0,$$

for some $w \in W$ and some x_j . For any $1 \leq i \leq \ell$, let L_i be the Levi subgroup containing T such that $\Delta(L_i) = \Delta \setminus \{\alpha_i\}$. By the assumption (11), $w^{-1}\lambda \in \sum_{\alpha_i \in \Delta(L_j)} \mathbb{Z}\alpha_i$. Moreover, we can choose $w \in W^{L_j}$ and hence $w^{-1}\lambda$ is a dominant weight for L_j . In particular, $v_{w^{-1}\lambda}$ is a highest weight vector for L_j , where $v_{w^{-1}\lambda}$ is a nonzero vector of (extremal) weight $w^{-1}\lambda$ in $V(\lambda)$. (To prove this, observe that $|w^{-1}\lambda + \alpha_i| > |\lambda|$ for any $\alpha_i \in \Delta(L_j)$, and hence $w^{-1}\lambda + \alpha_i$ can not be a weight of $V(\lambda)$.) Thus, the L_j -submodule $V_{L_j}(w^{-1}\lambda)$ of $V(\lambda)$ generated by $v_{w^{-1}\lambda}$ is an irreducible L_j -module. By [Ku, Lemma 3.1], applied to the L_j -module $V_{L_j}(w^{-1}\lambda)$, we get that $V_{L_j}(w^{-1}\lambda)$ contains the zero weight space. Hence, by [Ku, Lemma 3.4], there exists a $g \in L_j$ such that $gw^{-1}B \in X^{ss}(\lambda)$. Define the one parameter subgroup $\delta_j := \text{Exp}(\mathbb{Z}x_j)$. Then, $\mu^{\mathcal{L}(\lambda)}(gw^{-1}B, \delta_j) = \mu^{\mathcal{L}(\lambda)}(w^{-1}B, \delta_j)$, since g fixes δ_j . But, $\mu^{\mathcal{L}(\lambda)}(w^{-1}B, \delta_j) = 0$, by (5) (due to the assumption (11)). Thus, $gw^{-1}B \notin X^s(\lambda)$ by Proposition (3.3) (d). \square

For any $w \in W$ and $1 \leq i \leq \ell$, define the hyperplane

$$H_{w,i} := \{\lambda \in \Lambda(\mathbb{R}) : \lambda(wx_i) = 0\}.$$

Decompose into connected components:

$$\Lambda^{++}(\mathbb{R}) \setminus (\cup_{w \in W, 1 \leq i \leq \ell} H_{w,i}) = \sqcup_{k=1}^N C_k.$$

The following corollary follows immediately from Proposition (3.5) and [DH, Theorems 3.3.2 and 3.4.2].

Corollary (3.6). *With the notation as above, $\{C_1, \dots, C_N\}$ are precisely the GIT classes of maximal dimension (equal to $\dim \mathfrak{t}$).*

Lemma (3.7). *For any GIT class C_k (of maximal dimension) and any $\lambda \in \Gamma$, the line bundle $\mathcal{L}(\lambda)$ descends as a line bundle on the GIT quotient $X_T(C_k)$. We denote this line bundle by $\widehat{\mathcal{L}}_{C_k}(\lambda)$.*

Proof. By Theorem (3.1), for any $\lambda \in \Lambda^{++} \cap \Gamma$, the line bundle $\mathcal{L}(\lambda)$ on X descends to a line bundle on $X_T(\lambda)$. Hence, for any $\lambda \in \Gamma \cap C_k$, the line bundle $\mathcal{L}(\lambda)$ descends to a line bundle $\widehat{\mathcal{L}}_{C_k}(\lambda)$ on $X_T(C_k)$.

Let $\mathbb{Z}(\Gamma \cap C_k)$ denote the subgroup of Γ generated by the semigroup $\Gamma \cap C_k$. For any $\lambda = \lambda_1 - \lambda_2 \in \mathbb{Z}(\Gamma \cap C_k)$ (for $\lambda_1, \lambda_2 \in \Gamma \cap C_k$), define

$$\widehat{\mathcal{L}}_{C_k}(\lambda) = \widehat{\mathcal{L}}_{C_k}(\lambda_1) \otimes \widehat{\mathcal{L}}_{C_k}(\lambda_2)^*.$$

We now show that $\widehat{\mathcal{L}}_{C_k}(\lambda)$ is well defined, i.e., it does not depend upon the choice of the decomposition $\lambda = \lambda_1 - \lambda_2$ as above. Take another decomposition $\lambda = \lambda'_1 - \lambda'_2$, with $\lambda'_1, \lambda'_2 \in \Gamma \cap C_k$. Thus, $\lambda_1 + \lambda'_2 = \lambda'_1 + \lambda_2 \in \Gamma \cap C_k$ (since $\Gamma \cap C_k$ is a semigroup). In particular, $\widehat{\mathcal{L}}_{C_k}(\lambda_1 + \lambda'_2) \simeq \widehat{\mathcal{L}}_{C_k}(\lambda'_1 + \lambda_2)$.

But, from the uniqueness of $\widehat{\mathcal{L}}_{C_k}(\lambda)$ (cf. [T, § 3]), we have $\widehat{\mathcal{L}}_{C_k}(\lambda_1 + \lambda'_2) \simeq \widehat{\mathcal{L}}_{C_k}(\lambda'_1) \otimes \widehat{\mathcal{L}}_{C_k}(\lambda_2)$. This proves the assertion that $\widehat{\mathcal{L}}_{C_k}(\lambda)$ is well defined.

Observe that, by definition, C_k is an open convex cone in $\Lambda(\mathbb{R})$. We next claim that

$$(12) \quad \mathbb{Z}(\Gamma \cap C_k) = \Gamma.$$

Take a \mathbb{Z} -basis $\{\gamma_1, \dots, \gamma_\ell\}$ of Γ and let $d := \max_i \|\gamma_i\|$, with respect to a norm $\|\cdot\|$ on $\Lambda(\mathbb{R})$. Take a ‘large enough’ $\gamma \in \Gamma \cap C_k$ such that the closed ball $B(\gamma, d)$ of radius d centered at γ is contained in C_k . Then, for any $1 \leq i \leq \ell$, $\gamma + \gamma_i \in B(\gamma, d)$ and hence $\gamma, \gamma + \gamma_i \in \Gamma \cap C_k$ for any i . Thus, each $\gamma_i \in \mathbb{Z}(\Gamma \cap C_k)$ and hence $\Gamma = \mathbb{Z}(\Gamma \cap C_k)$, proving the assertion (12). Thus, the lemma is proved. \square

4. THE MAIN RESULT AND ITS PROOF

Let $\mu_0 : \Lambda^+ \rightarrow \mathbb{Z}_+$ be the function: $\mu_0 = \dim V(\lambda)_0$, where $V(\lambda)_0$ is the 0-weight space of $V(\lambda)$. Following the notation from Sections 2 and 3, the following is our main result.

Theorem (4.1). *Let G be a connected, adjoint, simple algebraic group. Let $\bar{\mu} = \mu + \Gamma$ be a coset of Γ in Q , where Γ is as in Theorem (3.1). Then, for any GIT class C_k (of maximal dimension), $1 \leq k \leq N$, there exists a polynomial $f_{\bar{\mu},k} : \Lambda(\mathbb{R}) \rightarrow \mathbb{R}$ with rational coefficients of degree $\leq \dim_{\mathbb{C}} X - \ell$, such that*

$$(13) \quad f_{\bar{\mu},k}(\lambda) = \mu_0(\lambda), \quad \text{for all } \lambda \in \bar{C}_k \cap \bar{\mu},$$

where \bar{C}_k is the closure of C_k inside $\Lambda(\mathbb{R})$. Further, $f_{\bar{\mu},k}$ has constant term 1.

Proof. By the Borel-Weil theorem, for any $\lambda \in \Lambda^+$,

$$(14) \quad \mu_0(\lambda) = \dim \left(H^0(X, \mathcal{L}(\lambda))^T \right),$$

since

$$\dim(V(\lambda)_0) = \dim((V(\lambda)^*)_0).$$

Moreover, by the Borel-Weil-Bott theorem, for $\lambda \in \Lambda^+$,

$$(15) \quad H^p(X, \mathcal{L}(\lambda)) = 0, \quad \text{for all } p > 0.$$

We first prove the theorem for $\lambda \in C_k \cap \bar{\mu}$:

Take $\lambda \in C_k \cap \bar{\mu}$. Let $\pi : X^{ss}(C_k) \rightarrow X_T(C_k)$ be the standard quotient map. For any T -equivariant sheaf \mathcal{S} on $X^{ss}(C_k)$, define the T -invariant direct image sheaf $\pi_*^T(\mathcal{S})$ as the sheaf on $X_T(C_k)$ with sections $U \mapsto \Gamma(\pi^{-1}(U), \mathcal{S})^T$. Then, by Lemma (3.7), and the projection formula for π_*^T ,

$$(16) \quad \pi_*^T(\mathcal{L}(\lambda)) \simeq \pi_*^T(\mathcal{L}(\mu)) \otimes \widehat{\mathcal{L}}_{C_k}(\lambda - \mu).$$

By [T, Remark 3.3(i)] and (15), we get

$$(17) \quad \begin{aligned} H^p \left(X_T(C_k), \pi_*^T(\mathcal{L}(\lambda)) \right) &\simeq H^0(X, \mathcal{L}(\lambda))^T, \quad \text{for } p = 0 \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Thus, for $\lambda \in C_k \cap \bar{\mu}$, by (14),

$$(18) \quad \mu_0(\lambda) = \chi \left(X_T(C_k), \pi_*^T(\mathcal{L}(\lambda)) \right),$$

where for any projective variety Y and a coherent sheaf \mathcal{S} on Y , we define the Euler-Poincaré characteristic

$$\chi(Y, \mathcal{S}) := \sum_{i \geq 0} (-1)^i \dim H^i(Y, \mathcal{S}).$$

Now, take a basis (as a \mathbb{Z} -module) $\{\gamma_1, \dots, \gamma_\ell\}$ of the lattice $\Gamma \subset \Lambda(\mathbb{R})$. Then, for any $\lambda = \mu + \sum_{i=1}^{\ell} a_i \gamma_i \in \bar{\mu}$, with $a_i \in \mathbb{Z}$, we have by (16),

$$(19) \quad \pi_*^T(\mathcal{L}(\lambda)) \simeq \pi_*^T(\mathcal{L}(\mu)) \otimes \widehat{\mathcal{L}}_{C_k} \left(\sum a_i \gamma_i \right).$$

Thus, by the Riemann-Roch theorem for singular varieties (cf. [F, Theorem 18.3]) applied to the sheaf $\pi_*^T(\mathcal{L}(\lambda))$, we get for any $\lambda = \mu + \sum a_i \gamma_i \in \bar{\mu}$,

$$(20) \quad \chi(X_T(C_k), \pi_*^T(\mathcal{L}(\lambda))) = \sum_{n \geq 0} \int_{X_T(C_k)} \frac{(a_1 c_1(\gamma_1) + \cdots + a_\ell c_1(\gamma_\ell))^n}{n!} \cap \tau(\pi_*^T(\mathcal{L}(\mu))),$$

where $\tau(\pi_*^T(\mathcal{L}(\mu)))$ is a certain class in the chow group $A_*(X_T(C_k)) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $c_1(\gamma_i)$ is the first Chern class of the line bundle $\widehat{\mathcal{L}}_{C_k}(\gamma_i)$. Combining (18) and (20), we get that for any $\lambda \in C_k \cap \bar{\mu}$, $\mu_0(\lambda)$ is a polynomial $f_{\bar{\mu},k}$ with rational coefficients in the variables $\{a_i\} : \lambda = \mu + \sum_{i=1}^{\ell} a_i \gamma_i$.

Since $X^s(C_k) \neq \emptyset$ by Lemma (3.4), $\dim(X_T(C_k)) = \dim X - \ell$. Thus, $\deg f_{\bar{\mu},k} \leq \dim X - \ell$. This proves the theorem for $\lambda \in C_k \cap \bar{\mu}$.

We now come to the proof of the theorem for any $\lambda \in \bar{C}_k \cap \bar{\mu}$:

Let $P = P_\lambda \supset B$ be the unique parabolic subgroup such that the line bundle $\mathcal{L}(\lambda)$ descends as an ample line bundle (denoted $\mathcal{L}^P(\lambda)$) on $X^P := G/P$ via the standard projection $q : G/B \rightarrow G/P$. Fix $\mu \in C_k \cap \Lambda$. By [T, §1.2], applied to $q : G/B \rightarrow G/P$, we get that $q^*(\mathcal{L}^P(\lambda))$ is adapted to the stratification on X induced from $q^*(\mathcal{L}^P(\lambda)) + \epsilon \mathcal{L}(\lambda)$, for any small rational $\epsilon > 0$ (cf. loc. cit. for the terminology). Thus, by [T, Theorem 3.2.a and Remarks 3.3], we get that (for any $\lambda \in \bar{C}_k \cap \bar{\mu}$)

$$(21) \quad \mu_0(\lambda) = \chi(X_T(C_k), \pi_*^T q^*(\mathcal{L}^P(\lambda))) = \chi(X_T(C_k), \pi_*^T(\mathcal{L}(\lambda))).$$

Hence, the identity (18) is established for any $\lambda \in \bar{C}_k \cap \bar{\mu}$. Thus, by the above proof, $\mu_0(\lambda) = f_{\bar{\mu},k}$, where $f_{\bar{\mu},k}$ is the polynomial given above.

By the formula (20), the constant term of $f_{\bar{\mu},k}$ is equal to

$$\chi(X_T(C_k), \pi_*^T(\mathcal{L}(0))),$$

which is 1 by the identity (21), since $\mu_0(0) = 1$. This completes the proof of the theorem. \square

Remark (4.2). (a) By a similar proof, we can obtain a piecewise polynomial behavior of the dimension of any weight space (of a fixed weight μ) in any finite dimensional irreducible representation $V(\lambda)$, by considering the GIT theory associated to the T -equivariant line bundle $\mathcal{L}(\lambda)$ twisted by the character μ^{-1} .

(b) By a similar proof, we can also obtain a piecewise polynomial behavior of the dimension of H -invariant subspace in any finite dimensional irreducible representation $V(\lambda)$ of G , where $H \subset G$ is a reductive subgroup. In this case, we will need to apply the GIT theory to the line bundle $\mathcal{L}(\lambda)$ itself but with respect to the group H . However, in this general case, we do

not have a precise description of the lattice Γ as in Theorem (3.1), nor do we have an explicit description of the GIT classes of maximal dimension as in Corollary (3.6).

(c) As pointed out by Kapil Paranjape, we can obtain the polynomial behaviour of $\chi(X_T(C_k), \pi_*^T(\mathcal{L}(\lambda)))$ as in the above proof (by using the Riemann-Roch theorem) more simply by applying Snapper's theorem (cf. [K, Theorem in Section 1]). However, the use of Riemann-Roch theorem gives a more precise result.

5. EXAMPLES OF A_2 AND B_2

In this section, we calculate the dimension of the T -invariant subspace in an irreducible representation of the rank 2 groups G of types A_2 and B_2 . In these cases, we can do the calculation via certain well-known branching laws to certain subgroups. But lacking any such general branching laws, we have not been able to handle G_2 .

We recall that irreducible representations of $\mathrm{GL}_{n+1}(\mathbb{C})$ are parametrized by their highest weights, which is an $(n+1)$ -tuple of integers:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \lambda_{n+1}.$$

It is a well-known theorem that an irreducible representation of $\mathrm{GL}_{n+1}(\mathbb{C})$ when restricted to $\mathrm{GL}_n(\mathbb{C})$ decomposes as a sum of irreducible representations with highest weights $(\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n)$ with

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_n \geq \mu_n \geq \lambda_{n+1},$$

and that these representations of $\mathrm{GL}_n(\mathbb{C})$ appear with multiplicity exactly one (cf. [GW, Theorem 8.1.1]).

Note that for an irreducible representation of $\mathrm{GL}_{n+1}(\mathbb{C})$ to have a nonzero zero weight space, it is necessary (and sufficient) for it to have trivial central character. For determining the zero weight space of a representation of $\mathrm{GL}_{n+1}(\mathbb{C})$ with trivial central character, it suffices to restrict it to $\mathrm{GL}_n(\mathbb{C})$ and consider those summands which have zero weight spaces for $\mathrm{GL}_n(\mathbb{C})$, and then to add these zero weight spaces of $\mathrm{GL}_n(\mathbb{C})$.

We calculate the dimension of the zero weight space of an irreducible representation of $\mathrm{GL}_3(\mathbb{C})$ by restricting the representation to $\mathrm{GL}_2(\mathbb{C})$, and noting that an irreducible representation of $\mathrm{GL}_2(\mathbb{C})$ parametrized by $(\mu_1 \geq \mu_2)$ has a nonzero weight space if and only if $\mu_1 + \mu_2 = 0$ and, in this case, the dimension of the zero weight space is 1. With these preliminaries, we leave the details of the straightforward proof of the following lemma to the reader.

Lemma (5.1). *An irreducible representation of $\mathrm{GL}_3(\mathbb{C})$ with highest weight $(\lambda_1 \geq \lambda_2 \geq \lambda_3)$, and with trivial central character, i.e., $\lambda_1 + \lambda_2 + \lambda_3 = 0$, has zero weight space of dimension*

- (1) $\lambda_1 - \lambda_2 + 1$, if $\lambda_2 \geq 0$, and
- (2) $\lambda_2 - \lambda_3 + 1$, if $\lambda_2 \leq 0$.

We next recall that irreducible representations of $\mathrm{SO}_{2n+1}(\mathbb{C})$ are parametrized by their highest weights, which is an n -tuple of integers with

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0.$$

Similarly, the irreducible representations of $\mathrm{SO}_{2n}(\mathbb{C})$ are parametrized by their highest weights, which is an n -tuple of integers with

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq |\lambda_n|.$$

It is a well-known theorem that an irreducible representation of $\mathrm{SO}_{2n+1}(\mathbb{C})$ when restricted to $\mathrm{SO}_{2n}(\mathbb{C})$ decomposes as a sum of irreducible representations with highest weights $(\mu_1 \geq \mu_2 \geq \cdots \geq |\mu_n|)$ with

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_n \geq |\mu_n|,$$

and that these representations of $\mathrm{SO}_{2n}(\mathbb{C})$ appear with multiplicity exactly one (cf. [GW, Theorem 8.1.3]).

We use this branching law from $\mathrm{SO}_5(\mathbb{C})$ to $\mathrm{SO}_4(\mathbb{C})$ to calculate the dimension of the zero weight space in an irreducible representation of $\mathrm{SO}_5(\mathbb{C})$. For this, we again note that the zero weight space in a $\mathrm{SO}_5(\mathbb{C})$ -representation is captured by those subrepresentations of $\mathrm{SO}_4(\mathbb{C})$ which have nonzero zero weight space. Further, note that $\mathrm{SO}_4(\mathbb{C})$ being the quotient of $\mathrm{SU}_2(\mathbb{C}) \times \mathrm{SU}_2(\mathbb{C})$ by the diagonal central element ± 1 , an irreducible representation of $\mathrm{SO}_4(\mathbb{C})$ has nonzero zero weight space if and only if its central character is trivial, and in this case the zero weight space is 1 dimensional. We also need to use the fact that the irreducible representation of $\mathrm{SO}_{2n}(\mathbb{C})$ parametrized by $(\lambda_1, \lambda_2, \dots, \lambda_n)$ has trivial central character if and only if $\lambda_1 + \lambda_2 + \cdots + \lambda_n$ is an even integer.

With these preliminaries, we leave the details of the straightforward proof of the following lemma to the reader.

Lemma (5.2). *An irreducible representation of $\mathrm{SO}_5(\mathbb{C})$ with highest weight $\lambda_1 \geq \lambda_2 \geq 0$ has zero weight space of dimension*

- (1) $(\lambda_1 - \lambda_2) \cdot \lambda_2 + \frac{\lambda_1 + \lambda_2}{2} + 1$, if $\lambda_1 + \lambda_2$ is an even integer.
- (2) $(\lambda_1 - \lambda_2) \cdot \lambda_2 + \frac{\lambda_1 + \lambda_2}{2} + \frac{1}{2}$, if $\lambda_1 + \lambda_2$ is an odd integer.

6. THE EXAMPLE OF PGL_4

In this section, we compute the dimension of the zero weight space of any irreducible representation of $G = \mathrm{PGL}_4(\mathbb{C})$.

Theorem (6.1). *For an irreducible representation of $\mathrm{GL}_4(\mathbb{C})$ with highest weight $(\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4)$, and with trivial central character, i.e., $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$, the dimension $d(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of the zero weight space is given as a piecewise polynomial in the domain $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ as follows:*

(1) $\lambda_2 \leq 0$, where it is given by the polynomial

$$p_1(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{1}{2}(\lambda_2 - \lambda_3 + 1)(\lambda_3 - \lambda_4 + 1)(\lambda_2 - \lambda_4 + 2).$$

(2) $\lambda_3 \geq 0$, where it is given by the polynomial

$$p_2(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{1}{2}(\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_1 - \lambda_3 + 2).$$

(3) $\lambda_2 > 0, \lambda_3 < 0, \lambda_1 + \lambda_4 \geq 0$, where it is given by the polynomial

$$p_3(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = -\frac{1}{2}(\lambda_1 + \lambda_2 + 2\lambda_3 + 1)(-\lambda_1\lambda_2 + 2\lambda_2^2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_3^2 - \lambda_1 + \lambda_3 - 2).$$

(4) $\lambda_2 > 0, \lambda_3 < 0, \lambda_1 + \lambda_4 \leq 0$, where it is given by the polynomial

$$p_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{1}{2}(-\lambda_1 + \lambda_2 - 1)(-\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + 3\lambda_3^2 - \lambda_1 - 2\lambda_2 - \lambda_3 - 2).$$

The automorphism $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \longrightarrow (-\lambda_4, -\lambda_3, -\lambda_2, -\lambda_1)$, which corresponds to taking a representation to its dual, interchanges the regions (1) and (2), and their polynomials, and similarly regions (3) and (4) and their polynomials. Further, we have

$$p_3 - p_4 = (\lambda_2 + \lambda_3) - (\lambda_2 + \lambda_3)^3.$$

Proof. The method we follow to prove this theorem is also based on the restriction of a $\mathrm{GL}_4(\mathbb{C})$ representation to $\mathrm{GL}_3(\mathbb{C})$, as we did in the previous section for the calculation of the zero weight space for $\mathrm{GL}_3(\mathbb{C})$ -representations. We start with an irreducible representation of $\mathrm{GL}_4(\mathbb{C})$ with highest weight $(\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4)$, and with trivial central character, i.e., $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$.

We look at irreducible representations of $\mathrm{GL}_3(\mathbb{C})$ with highest weight $(\mu_1 \geq \mu_2 \geq \mu_3)$ appearing in this representation of $\mathrm{GL}_4(\mathbb{C})$. Thus, we have

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4.$$

For analyzing the zero weight space, it suffices to consider only those representations of $\mathrm{GL}_3(\mathbb{C})$ with highest weight (μ_1, μ_2, μ_3) with $\mu_1 + \mu_2 + \mu_3 = 0$; it is actually keeping track of this central character condition (on $\mathrm{GL}_3(\mathbb{C})$) that complicates our analysis.

Denote the dimension of the zero weight space in the irreducible representation of $\mathrm{GL}_4(\mathbb{C})$ with highest weight $(\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4)$ by $d(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Similarly, denote the dimension of the zero weight space in the irreducible representation of $\mathrm{GL}_3(\mathbb{C})$ with highest weight (μ_1, μ_2, μ_3) by $d(\mu_1, \mu_2, \mu_3)$; we will always assume that the central character of this representation of $\mathrm{GL}_3(\mathbb{C})$ is trivial, and so $d(\mu_1, \mu_2, \mu_3)$ is a positive integer, explicitly given by Lemma 5.1. We remind the reader from loc. cit. that the value of $d(\mu_1, \mu_2, \mu_3)$ is a polynomial in (μ_1, μ_2, μ_3) (of degree 1) which depends on whether μ_2 is non-negative or non-positive.

Denote the interval $[\lambda_1, \lambda_2]$ by I_1 (we abuse the notation $[\lambda_1, \lambda_2]$ which is customarily denoted by $[\lambda_2, \lambda_1]$), the interval $[\lambda_2, \lambda_3]$ by I_2 , and the interval $[\lambda_3, \lambda_4]$ by I_3 . Our problem consists in choosing integers $\mu_i \in I_i$ such that $\mu_1 + \mu_2 + \mu_3 = 0$.

One lucky situation is when the set $I_j + I_k \subset -I_\ell$, for a triple $\{i, j, k\} = \{1, 2, 3\}$, in which case, one can choose $\mu_j \in I_j, \mu_k \in I_k$ arbitrarily, and then $\mu_\ell = -(\mu_j + \mu_k)$ automatically belongs to I_ℓ . This is what happens in cases I and II below; but the other cases that we deal with in III, ..., VI, the analysis is considerably more complicated.

Case I: $\lambda_2 \leq 0$, and therefore $\lambda_1 \geq 0 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$.

This implies that $\mu_1 \geq 0 \geq \mu_2 \geq \mu_3$; in particular, in this case μ_2 is always ≤ 0 . Further, $I_2 + I_3 = [\lambda_2 + \lambda_3, \lambda_3 + \lambda_4]$ is contained in $-I_1 = [-\lambda_2, -\lambda_1]$.

Therefore, by Lemma 5.1,

$$\begin{aligned}
 d(\lambda_1, \lambda_2, \lambda_3, \lambda_4) &= \sum_{\mu_i \in I_i} d(\mu_1, \mu_2, \mu_3) \\
 &= \sum_{\mu_i \in I_i} (\mu_2 - \mu_3 + 1) \\
 &= \sum_{\substack{\lambda_2 \geq \mu_2 \geq \lambda_3 \\ \lambda_3 \geq \mu_3 \geq \lambda_4}} (\mu_2 - \mu_3 + 1) \\
 &= \sum_{\lambda_3 \geq \mu_3 \geq \lambda_4} \left[\frac{(\lambda_2 + \lambda_3)(\lambda_2 - \lambda_3 + 1) - (\mu_3 - 1)(\lambda_2 - \lambda_3 + 1)}{2} \right] \\
 &= \frac{(\lambda_2 + \lambda_3)(\lambda_2 - \lambda_3 + 1)(\lambda_3 - \lambda_4 + 1)}{2} - \frac{(\lambda_2 - \lambda_3 + 1)(\lambda_3 + \lambda_4 - 2)(\lambda_3 - \lambda_4 + 1)}{2} \\
 &= \frac{1}{2}(\lambda_2 - \lambda_3 + 1)(\lambda_3 - \lambda_4 + 1)(\lambda_2 - \lambda_4 + 2).
 \end{aligned}$$

Case II: $\lambda_3 \geq 0$, and therefore $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0 \geq \lambda_4$.

This implies that $\mu_1 \geq \mu_2 \geq 0$; in particular, in this case μ_2 is always ≥ 0 . Further, $I_1 + I_2 = [\lambda_1 + \lambda_2, \lambda_2 + \lambda_3]$ is contained in $-I_3 = [-\lambda_4, -\lambda_3]$.

Therefore, by Lemma 5.1,

$$\begin{aligned}
 d(\lambda_1, \lambda_2, \lambda_3, \lambda_4) &= \sum_{\mu_i \in I_i} d(\mu_1, \mu_2, \mu_3) \\
 &= \sum_{\mu_i \in I_i} (\mu_1 - \mu_2 + 1) \\
 &= \sum_{\substack{\lambda_1 \geq \mu_1 \geq \lambda_2 \\ \lambda_2 \geq \mu_2 \geq \lambda_3}} (\mu_1 - \mu_2 + 1) \\
 &= \frac{1}{2}(\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_1 - \lambda_3 + 2).
 \end{aligned}$$

Rest of the cases: $\lambda_2 > 0 > \lambda_3$, and therefore $\lambda_1 \geq \lambda_2 > 0 > \lambda_3 \geq \lambda_4$.

Given that $\mu_i \in I_i$ with $\mu_1 + \mu_2 + \mu_3 = 0$, we find that

$$\lambda_3 \geq -(\mu_1 + \mu_2) \geq \lambda_4,$$

and therefore,

$$-\lambda_4 - \mu_2 \geq \mu_1 \geq -\lambda_3 - \mu_2.$$

Since we already have

$$\lambda_1 \geq \mu_1 \geq \lambda_2,$$

μ_1 is in the intersection of the two intervals $-\lambda_4 - \mu_2 \geq \mu_1 \geq -\lambda_3 - \mu_2$ and $\lambda_1 \geq \mu_1 \geq \lambda_2$. Therefore, μ_1 must belong to the interval

$$I(\mu_2) = [\min(\lambda_1, -\lambda_4 - \mu_2), \max(-\lambda_3 - \mu_2, \lambda_2)].$$

Conversely, it is clear that if $\mu_2 \in I_2$, $\mu_1 \in I(\mu_2)$, and $\mu_3 = -(\mu_1 + \mu_2)$, then each of the μ_i belongs to I_i , and $\mu_1 + \mu_2 + \mu_3 = 0$.

Thus, we can start the calculation of the dimension of the zero weight space in the representation of $\mathrm{GL}_4(\mathbb{C})$ with highest weight $(\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4)$, and with trivial central character as

$$\begin{aligned}
 d(\lambda_1, \lambda_2, \lambda_3, \lambda_4) &= \sum_{\mu_i \in I_i} d(\mu_1, \mu_2, \mu_3) \\
 &= \sum_{\substack{\lambda_2 \geq \mu_2 > 0 \\ \mu_1 \in I(\mu_2)}} (\mu_1 - \mu_2 + 1) + \sum_{\substack{0 \geq \mu_2 \geq \lambda_3 \\ \mu_1 \in I(\mu_2)}} (\mu_2 - \mu_3 + 1).
 \end{aligned}$$

At this point, we assume that $\lambda_1 + \lambda_4 \geq 0$. In this case, if $\mu_2 \geq 0$, then $\lambda_1 \geq -\lambda_4 - \mu_2$. On the other hand, under the same condition (i.e., $\lambda_1 + \lambda_4 \geq 0$), if $\mu_2 \leq 0$, then $-\lambda_3 - \mu_2 \geq \lambda_2$. This means that for $\mu_2 \geq 0$, $I(\mu_2) =$

$[\min(\lambda_1, -\lambda_4 - \mu_2), \max(-\lambda_3 - \mu_2, \lambda_2)] = [-\lambda_4 - \mu_2, \max(-\lambda_3 - \mu_2, \lambda_2)]$, and for $\mu_2 \leq 0$, $I(\mu_2) = [\min(\lambda_1, -\lambda_4 - \mu_2), -\lambda_3 - \mu_2]$. Therefore, we get

$$\begin{aligned}
d(\lambda_1, \lambda_2, \lambda_3, \lambda_4) &= \sum_{\mu_i \in I_i} d(\mu_1, \mu_2, \mu_3) \\
&= \sum_{\substack{\lambda_2 \geq \mu_2 > 0 \\ -\lambda_4 - \mu_2 \geq \mu_1 \geq \max(-\lambda_3 - \mu_2, \lambda_2)}} (\mu_1 - \mu_2 + 1) \\
&+ \sum_{\substack{0 \geq \mu_2 \geq \lambda_3 \\ \min(\lambda_1, -\lambda_4 - \mu_2) \geq \mu_1 \geq -\lambda_3 - \mu_2}} (\mu_1 + 2\mu_2 + 1).
\end{aligned}$$

At this point, we assume that besides $\lambda_1 + \lambda_4 \geq 0$, we also have $2\lambda_2 + \lambda_3 \geq 0$; this latter condition has the effect that the region $[\lambda_2, 0]$ where μ_2 is supposed to belong, splits into two regions where $\max(\lambda_3 - \mu_2, \lambda_2)$ takes the two possible options. Similarly, the region $[0, \lambda_3]$ where μ_2 belongs in the second sum gets divided into two regions.

Case III: $\lambda_1 + \lambda_4 \geq 0$ and $2\lambda_2 + \lambda_3 \geq 0$.

$$\begin{aligned}
d(\lambda_1, \lambda_2, \lambda_3, \lambda_4) &= \sum_{\mu_i \in I_i} d(\mu_1, \mu_2, \mu_3) \\
&= \sum_{\substack{\lambda_2 \geq \mu_2 \geq -(\lambda_2 + \lambda_3) \\ -\lambda_4 - \mu_2 \geq \mu_1 \geq \lambda_2}} (\mu_1 - \mu_2 + 1) + \sum_{\substack{-(\lambda_2 + \lambda_3) > \mu_2 > 0 \\ -\lambda_4 - \mu_2 \geq \mu_1 \geq -\lambda_3 - \mu_2}} (\mu_1 - \mu_2 + 1) \\
&+ \sum_{\substack{0 \geq \mu_2 \geq -(\lambda_1 + \lambda_4) \\ -\lambda_4 - \mu_2 \geq \mu_1 \geq -\lambda_3 - \mu_2}} (\mu_1 + 2\mu_2 + 1) + \sum_{\substack{-(\lambda_1 + \lambda_4) > \mu_2 \geq \lambda_3 \\ \lambda_1 \geq \mu_1 \geq -\lambda_3 - \mu_2}} (\mu_1 + 2\mu_2 + 1) \\
&= \frac{1}{4}[-4\lambda_1^3 - 2\lambda_1^2\lambda_2 - 2\lambda_2^3 - 6\lambda_1^2\lambda_3 + 4\lambda_1\lambda_2\lambda_3 - 4\lambda_2^2\lambda_3 - 4\lambda_1\lambda_3^2 - 2\lambda_3^3 - 6\lambda_1^2\lambda_4 - 4\lambda_1\lambda_2\lambda_4 \\
&\quad - 8\lambda_1\lambda_3\lambda_4 - 4\lambda_2\lambda_3\lambda_4 - 4\lambda_3^2\lambda_4 + 4\lambda_1\lambda_4^2 + 4\lambda_2\lambda_4^2 + 6\lambda_4^3 - 12\lambda_1^2 - 5\lambda_1\lambda_2 + \lambda_2^2 - 7\lambda_1\lambda_3 \\
&\quad + 8\lambda_2\lambda_3 + \lambda_3^2 - 34\lambda_1\lambda_4 - 15\lambda_2\lambda_4 - 13\lambda_3\lambda_4 - 20\lambda_4^2 + 5\lambda_1 + 3\lambda_2 + 5\lambda_3 - \lambda_4 + 4] \\
&= \frac{1}{4}[2\lambda_1^2\lambda_2 - 2\lambda_1\lambda_2^2 - 4\lambda_2^3 - 2\lambda_1^2\lambda_3 - 10\lambda_2^2\lambda_3 - 6\lambda_1\lambda_3^2 - 6\lambda_2\lambda_3^2 - 4\lambda_3^3 + 2\lambda_1^2 + 4\lambda_1\lambda_2 \\
&\quad - 4\lambda_2^2 - 4\lambda_2\lambda_3 - 6\lambda_3^2 + 6\lambda_1 + 4\lambda_2 + 6\lambda_3 + 4], \\
&= -\frac{1}{2}(\lambda_1 + \lambda_2 + 2\lambda_3 + 1)(-\lambda_1\lambda_2 + 2\lambda_2^2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_3^2 - \lambda_1 + \lambda_3 - 2),
\end{aligned}$$

where in the second last equality, we have used the equation $\lambda_4 = -(\lambda_1 + \lambda_2 + \lambda_3)$ to write the polynomial in only $\lambda_1, \lambda_2, \lambda_3$.

Case IV: $\lambda_1 + \lambda_4 \geq 0$ and $2\lambda_2 + \lambda_3 < 0$.

$$\begin{aligned}
 d(\lambda_1, \lambda_2, \lambda_3, \lambda_4) &= \sum_{\mu_i \in I_i} d(\mu_1, \mu_2, \mu_3) \\
 &= \sum_{\substack{\lambda_2 \geq \mu_2 > 0 \\ -\lambda_4 - \mu_2 \geq \mu_1 \geq -\lambda_3 - \mu_2}} (\mu_1 - \mu_2 + 1) \\
 &+ \sum_{\substack{0 \geq \mu_2 \geq -(\lambda_1 + \lambda_4) \\ -\lambda_4 - \mu_2 \geq \mu_1 \geq -\lambda_3 - \mu_2}} (\mu_1 + 2\mu_2 + 1) + \sum_{\substack{-(\lambda_1 + \lambda_4) > \mu_2 \geq \lambda_3 \\ \lambda_1 \geq \mu_1 \geq -\lambda_3 - \mu_2}} (\mu_1 + 2\mu_2 + 1) \\
 &= \frac{1}{4} [-2\lambda_1^3 - 2\lambda_1^2\lambda_2 - 2\lambda_1^2\lambda_3 + 2\lambda_1\lambda_2\lambda_3 - 4\lambda_2^2\lambda_3 - 2\lambda_1\lambda_3^2 - 2\lambda_2\lambda_3^2 - 2\lambda_3^3 - 4\lambda_1^2\lambda_4 - 4\lambda_1\lambda_2\lambda_4 \\
 &\quad + 4\lambda_2^2\lambda_4 - 4\lambda_1\lambda_3\lambda_4 - 2\lambda_2\lambda_3\lambda_4 - 2\lambda_3^2\lambda_4 + 2\lambda_2\lambda_4^2 - 2\lambda_3\lambda_4^2 + 2\lambda_4^3 - 5\lambda_1^2 - 3\lambda_1\lambda_2 - 4\lambda_2^2 + \\
 &\quad 3\lambda_2\lambda_3 + \lambda_3^2 - 14\lambda_1\lambda_4 - 7\lambda_2\lambda_4 - 7\lambda_4^2 + \lambda_1 - \lambda_2 + \lambda_3 - 5\lambda_4 + 4] \\
 &= \frac{1}{4} [2\lambda_1^2\lambda_2 - 2\lambda_1\lambda_2^2 - 4\lambda_2^3 - 2\lambda_1^2\lambda_3 - 10\lambda_2^2\lambda_3 - 6\lambda_1\lambda_3^2 - 6\lambda_2\lambda_3^2 - 4\lambda_3^3 + 2\lambda_1^2 + 4\lambda_1\lambda_2 \\
 &\quad - 4\lambda_2^2 - 4\lambda_2\lambda_3 - 6\lambda_3^2 + 6\lambda_1 + 4\lambda_2 + 6\lambda_3 + 4] \\
 &= -\frac{1}{2} (\lambda_1 + \lambda_2 + 2\lambda_3 + 1)(-\lambda_1\lambda_2 + 2\lambda_2^2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_3^2 - \lambda_1 + \lambda_3 - 2),
 \end{aligned}$$

where again in the second last equality, we have used the equation $\lambda_4 = -(\lambda_1 + \lambda_2 + \lambda_3)$ to write the polynomial in only $\lambda_1, \lambda_2, \lambda_3$.

Case V: $\lambda_1 + \lambda_4 < 0$ and $\lambda_2 + 2\lambda_3 \leq 0$.

$$\begin{aligned}
 d(\lambda_1, \lambda_2, \lambda_3, \lambda_4) &= \sum_{\mu_i \in I_i} d(\mu_1, \mu_2, \mu_3) \\
 &= \sum_{\substack{\lambda_2 \geq \mu_2 > (\lambda_2 + \lambda_3) \\ -\lambda_4 - \mu_2 \geq \mu_1 \geq \lambda_2}} (\mu_1 - \mu_2 + 1) + \sum_{\substack{(\lambda_2 + \lambda_3) \geq \mu_2 \geq 0 \\ \lambda_1 \geq \mu_1 \geq \lambda_2}} (\mu_1 - \mu_2 + 1) \\
 &+ \sum_{\substack{0 > \mu_2 > (\lambda_1 + \lambda_4) \\ \lambda_1 \geq \mu_1 \geq \lambda_2}} (\mu_1 + 2\mu_2 + 1) + \sum_{\substack{\lambda_1 + \lambda_4 \geq \mu_2 \geq \lambda_3 \\ \lambda_1 \geq \mu_1 \geq -\lambda_3 - \mu_2}} (\mu_1 + 2\mu_2 + 1) \\
 &= \frac{1}{4} [-2\lambda_1^3 + 2\lambda_1^2\lambda_2 + 2\lambda_1\lambda_2^2 + 2\lambda_2^3 - 2\lambda_1^2\lambda_3 + 2\lambda_1\lambda_2\lambda_3 + 2\lambda_2^2\lambda_3 - 4\lambda_1\lambda_3^2 + 4\lambda_2\lambda_3^2 + 4\lambda_1\lambda_2\lambda_4 \\
 &\quad + 2\lambda_2^2\lambda_4 + 4\lambda_1\lambda_3\lambda_4 - 2\lambda_2\lambda_3\lambda_4 + 4\lambda_1\lambda_4^2 + 2\lambda_2\lambda_4^2 + 2\lambda_3\lambda_4^2 + 2\lambda_4^3 - 7\lambda_1^2 + \lambda_2^2 - 7\lambda_1\lambda_3 \\
 &\quad + 3\lambda_2\lambda_3 - 4\lambda_3^2 - 14\lambda_1\lambda_4 - 3\lambda_3\lambda_4 - 5\lambda_4^2 + 5\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + 4] \\
 &= \frac{1}{4} [2\lambda_1^2\lambda_2 - 2\lambda_1\lambda_2^2 - 2\lambda_1^2\lambda_3 + 2\lambda_2^2\lambda_3 - 6\lambda_1\lambda_3^2 + 6\lambda_2\lambda_3^2 + 2\lambda_1^2 + 4\lambda_1\lambda_2 - 4\lambda_2^2 \\
 &\quad - 4\lambda_2\lambda_3 - 6\lambda_3^2 + 6\lambda_1 + 2\lambda_3 + 4] \\
 &= \frac{1}{2} (-\lambda_1 + \lambda_2 - 1)(-\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + 3\lambda_3^2 - \lambda_1 - 2\lambda_2 - \lambda_3 - 2).
 \end{aligned}$$

Case VI: $\lambda_1 + \lambda_4 < 0$ and $\lambda_2 + 2\lambda_3 \geq 0$.

$$\begin{aligned}
d(\lambda_1, \lambda_2, \lambda_3, \lambda_4) &= \sum_{\mu_i \in I_i} d(\mu_1, \mu_2, \mu_3) \\
&= \sum_{\substack{\lambda_2 \geq \mu_2 > (\lambda_2 + \lambda_3) \\ -\lambda_4 - \mu_2 \geq \mu_1 \geq \lambda_2}} (\mu_1 - \mu_2 + 1) + \sum_{\substack{(\lambda_2 + \lambda_3) \geq \mu_2 \geq 0 \\ \lambda_1 \geq \mu_1 \geq \lambda_2}} (\mu_1 - \mu_2 + 1) \\
&\quad + \sum_{\substack{0 > \mu_2 \geq \lambda_3 \\ \lambda_1 \geq \mu_1 \geq \lambda_2}} (\mu_1 + 2\mu_2 + 1) \\
&= \frac{1}{4} [-2\lambda_1^3 + 2\lambda_1^2\lambda_2 + 2\lambda_1\lambda_2^2 + 2\lambda_2^3 - 2\lambda_1^2\lambda_3 + 2\lambda_1\lambda_2\lambda_3 + 2\lambda_2^2\lambda_3 - 4\lambda_1\lambda_3^2 + 4\lambda_2\lambda_3^2 + 4\lambda_1\lambda_2\lambda_4 \\
&\quad + 2\lambda_2^2\lambda_4 + 4\lambda_1\lambda_3\lambda_4 - 2\lambda_2\lambda_3\lambda_4 + 4\lambda_1\lambda_4^2 + 2\lambda_2\lambda_4^2 + 2\lambda_3\lambda_4^2 + 2\lambda_4^3 - 7\lambda_1^2 + \lambda_2^2 - 7\lambda_1\lambda_3 \\
&\quad + 3\lambda_2\lambda_3 - 4\lambda_3^2 - 14\lambda_1\lambda_4 - 3\lambda_3\lambda_4 - 5\lambda_4^2 + 5\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + 4] \\
&= \frac{1}{4} [2\lambda_1^2\lambda_2 - 2\lambda_1\lambda_2^2 - 2\lambda_1^2\lambda_3 + 2\lambda_2^2\lambda_3 - 6\lambda_1\lambda_3^2 + 6\lambda_2\lambda_3^2 + 2\lambda_1^2 + 4\lambda_1\lambda_2 - 4\lambda_2^2 \\
&\quad - 4\lambda_2\lambda_3 - 6\lambda_3^2 + 6\lambda_1 + 2\lambda_3 + 4] \\
&= \frac{1}{2} (-\lambda_1 + \lambda_2 - 1)(-\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + 3\lambda_3^2 - \lambda_1 - 2\lambda_2 - \lambda_3 - 2).
\end{aligned}$$

□

Remark (6.2). (a) The fact that the answers in the cases III and IV above are the same (similarly in the cases V and VI) seems not obvious apriori before the final answer is calculated via a software.

(b) It is curious to note that the polynomials p_1 and p_3 are equal for $\lambda_2 = 0$, and the polynomials p_2 and p_3 are equal for $\lambda_3 = 0$, but the polynomials p_1 and p_3 are not equal for $\lambda_3 = 0$. This means that $d(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ which is a polynomial function in the interior of a polyhedral cone is not always given by the same polynomial on the boundary.

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